Problems and Solutions to PSTAT 170 Discussion Section 5/29/2007

The following problems are taken from Chapter 12 of Hull.

Problem 12.2: The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

Solution: Recall that the standard deviation for any time period $\triangle t$ is $\sigma\sqrt{\triangle t}$ where σ is the volatility measured in years. Therefore, if we assume that there are only 252 trading days, then the standard deviation for 1 trading day is $.3\sqrt{1/252} = .019$.

Problem 12.4: Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of \$50 when the current stock price is \$50, the risk free interest rate is 10% per annum, and the volatility is 30% per annum.

Solution: Once the price to European calls, and puts has been derived, the pricing is fairly trivial: you just plug in the parameter values into the pricing equation. Remember that the price to European calls and puts was derived in Chapter 12 assuming the stock price follows geometric Brownian motion. The price of a European put is

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1)$$

where,

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

and $N(\cdot)$ is the cumulative probability distribution function for a standardized normal distribution. Therefore, plugging in the values and using the table in your book to compute $N(\cdot)$ we get:

$$d_1 = \frac{\ln(50/50) + (.1 + .09/2).25}{.3\sqrt{.25}} = .2417$$

$$d_2 = \frac{\ln(50/50) + (.1 - .09/2).25}{.3\sqrt{.25}} = .0917$$

and

$$\begin{aligned} p &= 50N(-.0917)e^{-1\times.25} - 50N(-.2417) \\ &= 50 \times .4634 \times e^{-1\times.25} - 50 \times .4045 \\ &= 2.37 \end{aligned}$$

Problem 12.7: The stock price is currently \$40. Assume that the expected return from the stock is 15% and that its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a two-year period?

Solution: The rate of return for a stock is defined as

$$\frac{1}{T}\ln\left(\frac{S_T}{S_0}\right)$$

We know in the geometric Brownian motion world that $\ln(\frac{S_T}{S_0})$ is normally distributed with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$. That is,

$$\ln\left(\frac{S_T}{S_0}\right) \sim N((\mu - \sigma^2/2)T, \sigma^2 T)$$

Therefore,

$$\frac{1}{T}\ln\left(\frac{S_T}{S_0}\right) \sim N((\mu - \sigma^2/2), \sigma^2/T)$$

Therefore, in this problem

$$\frac{1}{T}\ln\left(\frac{S_T}{S_0}\right) \sim N(.11875, .25^2/2)$$

Problem 12.8: A stock price follow geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is 38%.

(a) What is the probability that a European call option on a stock with a strike price of \$40 and a maturity date in 6 months will be exercised?

Solution: We know that with geometric Brownian motion, in 6 months (.5 years), the stock price $S_{.5}$ will have the form

$$S_{.5} = 38e^{(.16-.35^2/2).5+.35\times\sqrt{.5}Z}$$

where $Z \sim N(0,1)$. Notice that the randomness of $S_{.5}$ only comes from the randomness of the standard normal random variable Z. Recall that the call option with strike price \$40 will only be exercised if $S_{.5} > 40$. The probability of this happening is simply,

$$\begin{aligned}
\mathbb{P}(S_{.5} > 40) &= \mathbb{P}(38e^{(.16-.35^2/2).5+.35\times\sqrt{.5}Z} > 40) \\
&= \mathbb{P}(Z > \frac{\ln(40/38) - (.16 - .35^2/2).5}{.35\sqrt{.5}}) \\
&= \mathbb{P}(Z > .008) \\
&= 1 - \mathbb{P}(Z \leq .008) \\
&= 1 - .5032 \\
&= .4968
\end{aligned}$$

where in the second line above we solve for Z because we know the distribution of Z.

(b) What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

Solution: This can be solved by using the information we just found by doing the following,

$$\begin{aligned}
\mathbb{P}(S_{.5} < 40) &= 1 - \mathbb{P}(S_{.5} > 40) \\
&= 1 - \mathbb{P}(S_{.5} \geq 40) \\
&= 1 - .4968 \\
&= .5032
\end{aligned}$$

Problem 12.9: Prove that, with the notation in the chapter, a 95% confidence interval for S_T is between

$$S_0 e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}} \text{ and } S_0 e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

Solution: To answer this we first recall the following fact,

$$\ln(S_T/S_0) \sim N((\mu-\sigma^2/2)T, \sigma^2 T)$$

Hence,

$$\ln(S_T) \sim N(\ln(S_0) + (\mu-\sigma^2/2)T, \sigma^2 T)$$

Therefore, the mean of $\ln(S_T)$ is $\ln(S_0)+(\mu-\sigma^2/2)T$ and the *standard deviation* of $\ln(S_T)$ is $\sigma\sqrt{T}$. Recalling that a 95% confidence interval means that the interval that is 1.96 standard deviations away from the mean, we get the confidence interval for $\ln(S_T)$ is $[\ln(S_0) + (\mu - \sigma^2/2)T - 1.96\sigma\sqrt{T}, \ln(S_0) + (\mu-\sigma^2/2)T + 1.96\sigma\sqrt{T}]$. Therefore, the confidence interval for S_T is

$$[S_0 e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}}, S_0 e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}]$$

Problem 12.11: Assume that a non-dividend paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a security that pays off a dollar amount equal to $\ln S_T$ at time T, where S_T denotes the value of the stock price at time T.

(a) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S_t.
Solution: Recall that in the risk neutral world we assume that the stock follows geometric Brownian motion with the risk-free rate r replacing the expected return μ. That is,

$$dS_t = rS_t dt + \sigma S_t dz_t$$

$$\Longrightarrow S_T = S_t e^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}Z}$$

where $Z \sim N(0,1)$. Therefore, the risk-neutral price is given by

$$\begin{aligned} P(S_t,t) &= e^{-r(T-t)}\mathbb{E}[\ln S_T|S_t] \\ &= e^{-r(T-t)}\mathbb{E}[\ln (S_t e^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}Z})] \\ &= e^{-r(T-t)}\mathbb{E}[\ln S_t + (r-\sigma^2/2)(T-t) + \sigma\sqrt{T-t}Z] \\ &= e^{-r(T-t)}\left(\ln S_t + (r-\sigma^2/2)(T-t)\right) \end{aligned}$$

(b) Confirm that the price found above satisfies the Black-Scholes PDE with appropriate boundary conditions.
Solution: The Black-Scholes PDE says that the price of the derivative $P(S_t,t)$ has to satisfy the following PDE,

$$\frac{\partial P(S_t,t)}{\partial t} + rS_t\frac{\partial P(S_t,t)}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 P(S_t,t)}{\partial S_t^2} = rP(S_t,t)$$

Therefore, we need to calculate the three partial derivatives that appear in the PDE based on the price that we found in the previous problem, i.e.,

$$P(S_t,t) = e^{-r(T-t)}\left(\ln S_t + (r-\sigma^2/2)(T-t)\right)$$

Therefore,

$$\frac{\partial P(S_t,t)}{\partial t} = re^{-r(T-t)}\left(\ln S_t + (r-\frac{\sigma^2}{2})(T-t)\right) - e^{-r(T-t)}(r-\frac{\sigma^2}{2})$$

$$\frac{\partial P(S_t,t)}{\partial S_t} = \frac{e^{-r(T-t)}}{S_t}$$

$$\frac{\partial^2 P(S_t,t)}{\partial S_t^2} = -\frac{e^{-r(T-t)}}{S_t^2}$$

Therefore, substituting into the left hand side (LHS) of the PDE we have,

$$\begin{aligned} LHS &= e^{-r(T-T)}\left(\ln S_t + r(r-\sigma^2/2)(T-t) - (r-\sigma^2/2) + r - \sigma^2/2\right) \\ &= re^{-r(T-t)}\left(\ln S_t + (r-\sigma^2/2)(T-t)\right) \\ &= rP(S_t,t) \end{aligned}$$

where we also have the boundary condition that $P(S_T, T) = \ln S_T$. Hence, the price we found previously does indeed satisfy the Black-Scholes PDE.