Solutions to Homework #7

1. In this problem,
$$u = 55/50 = 1.1$$
$$d = 45/50 = .9$$
Therefore, the probability that the stock goes up is given by
$$p = \frac{e^{.1\times.5} - .9}{1.1 - .9} = 0.7564$$
Therefore, in 6 months, the put option pays out $\max(50-55, 0) = 0$ with probability .7564 and $\max(50-45, 0) = 5$ with probability 0.2436. Hence, the discounted expected value of the put option is
$$(0 \times 0.7564 + 5 \times 0.2436)e^{-0.1\times.5} = 1.16$$
Therefore, the price of the put is \$1.16.

2. In this problem, $S_0 = 50$ and over each of the next two three-month periods it is expected to go up by 6% or down by 5%. We have to find the value of a 6-month European call option with $K = 51$ and $r = .05$. Therefore, $u = 1.06$ and $d = .95$ and p is given by,
$$p = \frac{e^{.05\times3/`12} - .95}{1.06 - .95} = 0.5689$$
Recall that this is the p we use on each node of the tree so in it's computation we must use the δt that corresponds to the length of time interval in one period of the tree.

   After 6 months the possible values of the stock price are $50 \times u^2 = 56.18$ with probability p^2, $50 \times u \times d = 50.35$ with probability $2 \times p \times (1-p)$ and $50 \times d^2 = 45.125$ with probability $(1-p)^2$. Therefore, the possible payoff values of the European call are $\max(56.18 - 51, 0) = 5.18$ with probability p^2, $\max(50.35 - 51, 0) = 0$ with probability $2p \times (1-p)$, and $\max(45.125 - 51, 0)$ with probability $(1-p)^2$. Therefore, the price is
$$5.18 \times 0.5689^2 e^{-0.05\times.5} = 1.635$$

3. To find the price of a European put option in # 2 above with the same maturity and strike price, we note that the possible payoffs are $\max(51 - 56.18, 0) = 0$ with probability p^2, $\max(51-50.35, 0) = 0.65$ with probability $2p \times (1-p)$, and $\max(51-45.125, 0) = 5.875$ with probability $(1-p)^2$. Hence, the price is given by
$$\big(0.65 \times 2 \times 0.5689 \times 0.4311 + 5.875 \times 0.4311^2)e^{-0.05\times0.5}\big) = 1.376$$

To verify that the call price we found and the put price we found satisfies put-call parity, we note that

$$p + S_0 = 1.376 + 50 = 51.376$$

and

$$c + Ke^{-rT} = 1.635 + 51e^{-0.05\times .05} = 51.376$$

4. Consider a variable S that follows the process

$$dS = \mu dt + \sigma dz_t$$

For the first three years, $\mu = 2$ and $\sigma = 3$; for the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value of the variable is 5, what tis the probability distribution of the value of the variable at the end of year 6?

To solve the above problem, you need to find the value of S_6 in terms of S_3. Solving the above equation we have,

$$S_3 = S_0 + \mu t + \sigma z_3 = 5 + 2(3) + 3z_3 = 11 + 3z_3$$

where $z_3 \sim N(0, 3)$ random variable. Once we have S_3, we solve for S_6 in terms of S_3, that is,

$$S_6 = S_3 + \mu(6 - 3) + \sigma(z_6 - z_3) = (11 + 3z_3) + 3(3) + 4(z_6 - z_3)$$

Hence,

$$S_6 = 20 + 3z_3 + 4(z_6 - z_3)$$

where $z_3 \sim N(0, 3)$ and $z_6 - z_3 \sim N(0, 6 - 3 = 3)$. Also, we know that z_3 and $z_6 - z_3$ are independent. Therefore, using the property of sums of independent normal random variables,

$$S_6 \sim N(20, 3^2(3) + 4^2(3) = 75)$$

5. Suppose that G is a function of a stock price, S, and time. Suppose that σ_S and σ_G are the volatilities of S and G. Show that when the expected return of S increases by $\lambda\sigma_S$, the growth rate of G increases by $\lambda\sigma_G$, where λ is a constant.

We first apply Ito's lemma to $G(S_t, t)$. Therefore,

$$dG = \left(\frac{\partial G(S_t, t)}{\partial S}\mu S_t + \frac{\partial G(S_t, t)}{\partial t} + \frac{1}{2}\frac{\partial^2 G(S_t, t)}{\partial S^2}\sigma^2 S_t^2\right) dt + \frac{\partial G(S_t, t)}{\partial S}\sigma S_t dz_t$$

Therefore, the volatility σ_G is given by $\sigma_G = \frac{\partial G(S_t,t)}{\partial S}\sigma$. The growth rate of S_t is μ, and $\sigma_S = \sigma$ so when μ increases to $\mu + \lambda\sigma_S$ we increase the return of G (found by substituting $\mu + \lambda\sigma$ for μ in the equation for G) by

$$\frac{\partial G(S_t,t)}{\partial S}\lambda\sigma = \lambda\sigma_G$$

6. Suppose that a stock price, S_t, follows geometric Brownian motion with expected return μ and volatility σ. What is the process followed by the variable S_t^n?

   We apply Ito's lemma to S_t^n. Therefore,

$$\begin{aligned} dS_t^n &= \left(nS_t^{n-1}\mu S_t + \frac{1}{2}n(n-1)S_n^{n-2}\sigma^2 S_t^2\right)dt + nS_t^{n-1}\sigma S_t dz_t \\ &= \left(n\mu + \frac{1}{2}n(n-1)\sigma^2\right)S_t^n dt + n\sigma S_t^n dz_t \end{aligned}$$

   You should recognize that the above is the dynamics for geometric Brownian motion with drift $(n\mu + \frac{1}{2}n(n-1)\sigma^2)$ and volatility $n\sigma$. The solution is given by,

$$S_t^n = S_0^n e^{n(\mu-\sigma^2/2)t+n\sigma z_t}$$

7. Suppose that x_t is the yield to maturity with continuous compounding on a zero-coupon bond that pay \$1 at time T. Assume that x_t follows the pocess,

$$dx_t = a(x_0 - a)dt + sx_t dz_t$$

   where a, x_0, and s are positive constants and dz_t is a Wiener process. What is the process followed by the bond price?

   The price of the bond, given its yield x_t is

$$B(x_t,t) = e^{-(T-t)x_t}$$

   Using Ito's lemma we will calculate $dB(x_t,t)$

$$\begin{aligned} \frac{\partial B(x_t,t)}{\partial t} &= x_t e^{-(T-t)x_t} = x_t B(x_t,t) \\ \frac{\partial B(x_t,t)}{\partial x} &= -(T-t)e^{-(T-t)x_t} = -(T-t)B(x_t,t) \\ \frac{\partial^2 B(x_t,t)}{\partial x^2} &= (T-t)^2 e^{-(T-t)x_t} = (T-t)^2 B(x_t,t) \end{aligned}$$

The important observation to make about the above quantities is that we write all of the partials in terms of the process itself.

Therefore, using Ito's lemma,

$$dB = \left(-a(x_0 - x_t)(T-t) + x_t + \frac{1}{2}s^2 x_t^2 (T-t)^2 \right) B(x_t, t)dt - s x_t (T-t) B(x_t, t) dz_t$$