Derivation of the price of a European call option

To derive the price of the European call option of a non-dividend paying stock, one could solve the Black-Scholes PDE with appropriate boundary conditions, or calculate the expectation explicitly using techniques from probability. We will take the later approach as it is more straight forward and the understanding of the derivation will help prepare you for your final exam. Recall that we assume the dynamics of the stock price is given by geometric Brownian motion, and under the risk-neutral measure has dynamics:

$$dS_t = rS_t dt + \sigma S_t dz_t$$

where r is the risk-free rate. Furthermore, the solution to the above equation is given by

$$S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma Z_T}$$

where $Z_T \sim N(0,T)$.

The price of a European call option, with strike price K and maturity T is given by the expected value of the payoff of the option discounted back to today. That is,

$$c = e^{-rT}\mathbb{E}[\max(S_T - K, 0)] \tag{1}$$

We will now evaluate the above expectation to derive explicitly the price c. There are a couple of items that should be pointed out before we begin:

- $\max(S_T - K, 0) = (S_T - K)\mathbf{1}_{S_T \geq K}$. That is, to calculate the expectation of $\max(S_T - K, 0)$, all we have to do is integrate $(S_T - K)$ over the values $S_T \geq K$.
- Inside the expectation we will replace Z_T with $\sqrt{T}Z$ where $Z \sim N(0,1)$.
- The only randomness will be from Z, therefore, we simplify the expectation into terms that only involve Z and use the properties of the distribution of Z (i.e. that it's standard normal).

Hence, we do the following,

$$
\begin{aligned}
e^{-rT}\mathbb{E}[\max(S_T-K,0)] &= e^{-rT}\mathbb{E}[\max(S_0e^{(r-\frac{\sigma^2}{2})T+\sigma Z_T}-K,0)] \\
&= e^{-rT}\mathbb{E}[\max(S_0e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z}-K,0)] \\
&= e^{-rT}\int_{S_0e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z}\geq K}(S_0e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z}-K)\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&\qquad \text{integrate against the density of a N(0,1) R.V.} \\
&= e^{-rT}\int_{z\geq\frac{\ln(K/S_0)-(r-\sigma^2/2)T}{\sigma\sqrt{T}}}(S_0e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z}-K)\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&= e^{-rT}\int_{\frac{\ln(K/S_0)-(r-\sigma^2/2)T}{\sigma\sqrt{T}}}^{\infty}S_0e^{(r-\sigma^2/2)T+\sigma\sqrt{T}z}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&\quad -Ke^{-rT}\int_{\frac{\ln(K/S_0)-(r-\sigma^2/2)T}{\sigma\sqrt{T}}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&\qquad \text{figure out the bounds and split up the subtraction} \\
&= S_0e^{-\sigma^2/2T}\int_{-\infty}^{\frac{\ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}}\frac{1}{\sqrt{2\pi}}e^{-z^2/2+\sigma\sqrt{T}z}dz \\
&\quad -Ke^{-rT}\int_{-\infty}^{\frac{\ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&\qquad \text{change limits using property of normal density and simplify} \\
&= S_0\int_{-\infty}^{\frac{\ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}}\frac{1}{\sqrt{2\pi}}e^{-(z-\sigma\sqrt{T})^2/2}dz \\
&\quad -Ke^{-rT}\int_{-\infty}^{\frac{\ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&\qquad \text{complete the square and simplify} \\
&= S_0\int_{-\infty}^{\frac{\ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&\quad -Ke^{-rT}\int_{-\infty}^{\frac{\ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&\text{make the change of variables } x=z-\sigma\sqrt{T} \text{ and change the limits of integration} \\
&= S_0N(d_1)-Ke^{-rT}N(d_2)
\end{aligned}
$$

where,

$$d_1 = \frac{\ln{(S_0/K)} + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln{(S_0/K)} + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

The price of a European put can be found similarly or by using put-call parity.