Explanation and Derivation of the Black-Scholes PDE

The purpose of this document is to provide you with a clear explanation and derivation of the Black-Scholes PDE. Understanding the steps involved in the derivation will greatly aid you on the final exam. Therefore, we will cover everything step by step.

The purpose of the Black-Scholes PDE is to describe how the price of a derivative (option) evolves through time and through changes in the underlying security price.

As you have seen in all of the previous material, the key idea is risk, and where the risk is coming from. If we assume the stock price follows geometric Brownian motion,

$$dS_t = \mu S_t d_t + \sigma S_t dz_t$$

then all the risk in the stock price is coming from the dz_t term. Mimicking what we did in the binomial tree model, our goal will be to create a portfolio of 1 shorted option and a certain number of shares of stock such that all of the risk is eliminated in the portfolio. The price (current value) of the option is written as $P(S_t, t)$ and the number of shares we own at time t is written as $\triangle_t$. Therefore, we consider the following portfolio

$$\Pi = \triangle_t S_t - P(S_t, t) \tag{1}$$

We would like to make the above portfolio riskless. This means that we need to compute the full derivative of Π and try to solve for the value of $\triangle_t$ that eliminates the dz_t. The full derivative of Π is,

$$d\Pi \quad = \quad \triangle_t dS_t - dP(S_t, t)$$

Now, in the above expression, we know what dS_t is because we have assumed the dynamics of S_t is geometric Brownian motion. To compute $dP(S_t, t)$ we have to apply Ito's lemma. Applying Ito's lemma to $P(S_t, t)$ we find that,

$$dP(S_t,t) = \left(\frac{\partial P(S_t,t)}{\partial S} \mu S_t + \frac{\partial P(S_t,t)}{\partial t} + \frac{1}{2} \frac{\partial^2 P(S_t,t)}{\partial S^2} \sigma^2 S_t^2 \right) dt + \frac{\partial P(S_t,t)}{\partial S} \sigma S_t dz_t$$

Now, substituting the expressions we have for dS_t and $dP(S_t, t)$ into $d\Pi$ we find,

$$\begin{aligned} d\Pi \quad &= \quad \triangle_t \left(\mu S_t d_t + \sigma S_t dz_t \right) \\ &\quad - \left(\frac{\partial P(S_t,t)}{\partial S} \mu S_t + \frac{\partial P(S_t,t)}{\partial t} + \frac{1}{2} \frac{\partial^2 P(S_t,t)}{\partial S^2} \sigma^2 S_t^2 \right) dt - \frac{\partial P(S_t,t)}{\partial S} \sigma S_t dz_t \\ &= \quad \left(\triangle_t \mu S_t - \frac{\partial P(S_t,t)}{\partial S} \mu S_t - \frac{\partial P(S_t,t)}{\partial t} - \frac{1}{2} \frac{\partial^2 P(S_t,t)}{\partial S^2} \sigma^2 S_t^2 \right) dt \\ &\quad + \left(\triangle_t \sigma S_t - \frac{\partial P(S_t,t)}{\partial S} \sigma S_t \right) dz_t \end{aligned}$$

Recall that our goal was to find the value of $\triangle_t$ that eliminates the dz_t term in $d\Pi$; thereby eliminating the risk. By looking at the last equation above, you can see that if we set

$$\triangle_t = \frac{\partial P(S_t, t)}{\partial S}$$

then the dz_t term is eliminated! For now on we will assume that $\triangle_t = \frac{\partial P(S_t,t)}{\partial S}$. Choosing this particular number of shares to own is known as *delta* hedging and holding a delta-neutral portoflio. It eliminates the risk of linear shifts in the change in S_t. More complicated hedging, known as *gamma* hedging, will capture the curvature of the risk. It should be noted that delta-hedging only works for fairly small shifts in S_t and the portfolio must be rebalanced often to remain delta-neutral.

Now, back to the PDE. If we substitute the value of $\triangle_t$ we found into our expression for $d\Pi$ we get that

$$d\Pi = \left(-\frac{\partial P(S_t, t)}{\partial t} - \frac{1}{2} \frac{\partial^2 P(S_t, t)}{\partial S^2} \sigma^2 S_t^2 \right) dt$$

Now, since the portfolio is riskless, for there to be no-arbitrage the portfolio must grow at the riskless rate, i.e.,

$$d\Pi = r\Pi dt = r \left(\frac{\partial P(S_t, t)}{\partial S} S_t - P(S_t, t) \right) dt$$

Equating the two derivatives to one another, and simplifying, we get

$$\frac{\partial P(S_t, t)}{\partial t} + rS_t \frac{\partial P(S_t, t)}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 P(S_t, t)}{\partial S^2} = rP(S_t, t)$$

This is the Black-Scholes PDE. Different options will give different prices through specifying a particular boundary condition. For example, specifying $P(S_T, T) = \max(S_T - K, 0)$ means the option is a European call, whereas specifying $P(S_T, T) = \max(K - S_T, 0)$ means the option is a European put.

Also, it should be noted that even though we assumed the expected return on the stock was μ, that through the process of creating the riskless portfolio, this parameter was eliminated and the price only depends on the risk-free rate r and the volatility levelσ. This means that even though people might disagree on the expected growth rate of the stock, they will agree on its price (in the Black-Scholes world).